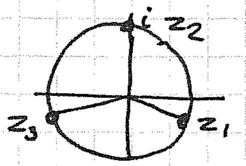
Tentamen Complexe Analyse 02/02/09

1. $\int_\gamma f(z)\,dz \overset{\text{def}}{=} \int_{t=a}^{b} f(\gamma(t))\,\gamma'(t)\,dt \Rightarrow \left|\int_\gamma f(z)\,dz\right| \leq \int_{t=a}^{b} \underbrace{|f(\gamma(t))|}_{\leq M}\,|\gamma'(t)|\,dt$

$\leq M \int_a^b |\gamma'(t)|\,dt = M\int_a^b \sqrt{u'(t)^2+v'(t)^2}\,dt = ML$

$\int_{|z|=r} \frac{1}{z^3+i}\,dz, \quad r>1, \quad |z^3+i| \geq \left||z|^3-|i|\right| = |r^3-1| = r^3-1 \Rightarrow \left|\int_{|z|=r}\frac{dz}{z^3+i}\right| \leq \frac{2\pi r}{r^3-1}$

In feite $\int_{|z|=r} \frac{dz}{z^3+i} = 0$



$2\pi i\left[\mathrm{Res}_{z_1} + \mathrm{Res}_{z_2} + \mathrm{Res}_{z_3}\right]$

$= 2\pi i\left[\frac{1}{3z_1^2} + \frac{1}{3z_2^2} + \frac{1}{3z_3^2}\right]$

$= \frac{2\pi i}{3}\left[\frac{1}{z_1^2} + \frac{1}{z_2^2} + \frac{1}{z_3^2}\right] = \frac{2\pi}{3}\left[z_1+z_2+z_3\right] = 0$

2. $f(z) = \frac{1}{z}e^{z^3}$ analytisch op $1 \leq |z| \leq 2$, niet const

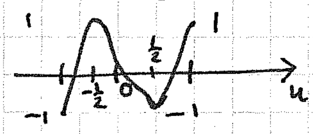
$\Rightarrow |f(z)|$ bezit max op $|z|=1$ of op $|z|=2$ $\qquad |f(z)|_{|z|=r} \leq \frac{1}{r}e^{r^3} = |f(r)|$!

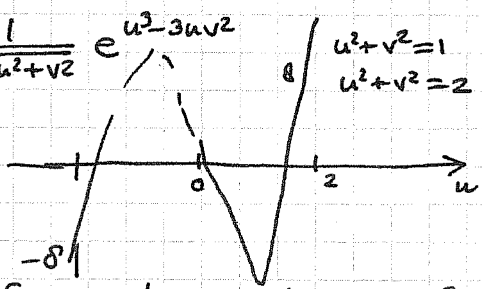
$r=1$ max. e $\qquad r=2$ max. $\frac{e^8}{2}$ $(>e)$

Anders: $|f(z)| = \frac{1}{\sqrt{u^2+v^2}}\,e^{u^3-3uv^2}$

$u^2+v^2=1 \quad u^3-3uv^2 = 4u^3-3u = 4u\left(u^2-\frac{3}{4}\right)$

$u^2+v^2=2 \quad u^3-3uv^2 = 4u^3-12u = 4u\left(u^2-3\right)$

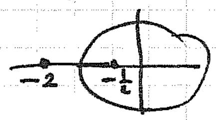




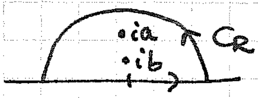
3. $\int_0^{2\pi} \frac{d\theta}{5+4\cos\theta} \overset{z=e^{i\theta}}{=} \int_{|z|=1} \frac{1}{5+4\,\frac{z+\frac{1}{z}}{2}}\,\frac{dz}{iz} = \frac{1}{i}\int_{|z|=1}\frac{1}{2z^2+5z+2} = \frac{1}{2i}\int_{|z|=1}\frac{dz}{z^2+\frac{5}{2}z+1}$

$z^2+\frac{5}{2}z+1=0 \Rightarrow z_1=-\frac{1}{2},\ z_2=-2$

$= \frac{2\pi i}{2i}\,\mathrm{Res}_{z_1}\frac{1}{z^2+\frac{5}{2}z+1} = \frac{2\pi i}{2i}\cdot\frac{1}{2z+\frac{5}{2}}\Big|_{z=z_1} = \frac{2\pi}{3}$



4. $\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}$

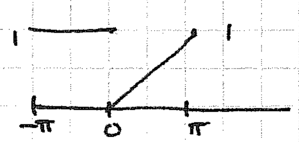


$\left|\int_{C_R}\cdots\,dz\right| \leq \frac{\pi R}{(R^2-a^2)(R^2-b^2)} \to 0,\quad R\to\infty$

$\int_{-R}^{R}\frac{e^{ix}}{(x^2+a^2)(x^2+b^2)}\,dx + \underbrace{\int_{C_R}\frac{e^{iz}}{(z^2+a^2)(z^2+b^2)}\,dz}_{\to 0} = 2\pi i\left[\mathrm{Res}_{ia} + \mathrm{Res}_{ib}\right]$

$= 2\pi i\left[\frac{e^{-a}}{b^2-a^2}\,\frac{1}{2ia} + \frac{e^{-b}}{a^2-b^2}\cdot\frac{1}{2ib}\right] = \frac{\pi}{a^2-b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$

5.



$F(t)$ 2π-periodiek, continu behalve in $t=0$ met sprong $\frac{1}{2}$

$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)\,dt = \frac{1}{2\pi}\left[\pi + \frac{1}{2}\pi\right] = \frac{3}{4}$

$n\neq 0 \quad 2\pi c_n = \int_{-\pi}^{0} e^{-int}\,dt + \int_0^{\pi}\frac{t}{\pi}e^{-int}\,dt = \frac{e^{-int}}{-in}\Big|_{-\pi}^{0} + \frac{t}{\pi}\,\frac{e^{-int}}{-in}\Big|_0^{\pi} - \int_0^{\pi}\frac{1}{\pi}\,\frac{e^{-int}}{-in}\,dt$

$= \frac{1}{-in}\left(1 - e^{in\pi} + e^{-in\pi}\right) - \frac{1}{\pi}\,\frac{e^{-int}}{(-in)^2}\Big|_0^{\pi} = \frac{i}{n} - \frac{1}{\pi n^2}\left(1-(-1)^n\right)$

$\lim_{N\to\infty}\sum_{-N}^{N} c_n e^{int} = F(t) \quad t\neq 0 \qquad \lim_{N\to\infty}\sum_{-N}^{N} c_n = \frac{1}{2}$

$c_0 + \sum_{-N}^{N}{}' c_n e^{int} = c_0 + \sum_{-N}^{N}{}'\left[\frac{i}{2\pi n} - \frac{1}{2\pi^2 n^2}\left(1-(-1)^n\right)\right](\cos nt + i\sin nt)$

$= \frac{3}{4} + \sum_{-N}^{N}{}'\left[\frac{-\sin nt}{2\pi n} - \frac{1}{2\pi^2 n^2}\left(1-(-1)^n\right)\cos nt\right] = \frac{3}{4} + \sum_{n=1}^{N}\frac{-2\sin nt}{2\pi n} - \sum_{n=1}^{N}\frac{2}{2\pi^2 n^2}\left(1-(-1)^n\right)\cos nt$

N.B. Voor $t=0$ volgt: $\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$